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# The Strongly Implicit Procedure for Biharmonic Problems

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A strongly implicit procedure is described which solves the system of 13 point finite difference equations associated with the biharmonic and similar fourth order elliptic equations. No factorization of the equation is required, and for the majority of problems, a universal set of iteration parameters provide rapid rates of convergence. In a comparison with another solution procedure for the biharmonic equation, the new method appears to reduce the computation required to about one-third.

## 1. INTRODUCTION

The numerical methods for solving systems of finite difference equations associated with partial differential equations have developed considerably in the last few years, due mainly to the increasing use of high speed digital computers. From the straightforward point iterative methods, such as Gauss–Seidel and successive over relaxation, through to line iterative methods, such as successive line over relaxation and the semiiterative Peaceman–Rachford alternating direction implicit method, we can trace an increasing use of partially implicit methods of solution. For these methods, in place of updating the values at each node (i.e., solving each equation) in turn, as for the point iterative methods, one updates all the points (i.e., simultaneously solves all the equations) belonging to a particular subset of the totality of points (equations). One then passes to the next subset. The subsets are, in general, disjoint and the totality of all the subsets is the whole set. The strongly implicit procedure devised by Stone [4] uses only one subset, namely the whole set of points, but in place of solving the whole set of finite difference equations as given, the method solves a very closely related system of equations which is generated from the original set so that it can readily be solved very simply by triangular decomposition, and so that the limit of the iterative sequence is the solution of the original system.

As originally proposed by Stone, the strongly implicit procedure was formulated for second order partial differential equations. A closely related procedure will be

described which has been devised for solving fourth order elliptic[1] (or parabolic) partial differential equations which yield systems of finite difference equations with 13 terms, such as the biharmonic equation. The method evolves as a fast procedure producing accurate results.

## 2. Outline of Method

We seek the solution of the system of $MN$ linear algebraic equations

$$\begin{aligned} &A(i,j)\,T(i,j-2) + B1(i,j)\,T(i-1,j-1) + B2(i,j)\,T(i,j-1) \\ &+ B3(i,j)\,T(i+1,j-1) + C1(i,j)\,T(i-2,j) + C2(i,j)\,T(i-1,j) \\ &+ C3(i,j)\,T(i,j) + C4(i,j)\,T(i+1,j) + C5(i,j)\,T(i+2,j) \\ &+ D1(i,j)\,T(i-1,j+1) + D2(i,j)\,T(i,j+1) + D3(i,j)\,T(i+1,j+1) \\ &+ E(i,j)\,T(i,j+2) = Q(i,j), \end{aligned}$$

with $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, N$, written in matrix form as

$$\mathbf{AT} = \mathbf{Q},$$

where $\mathbf{A}$ is a square matrix of order $MN$, $\mathbf{T}$ is a column vector of the elements $T(1, 1), T(2, 1), \ldots, T(M, 1), T(1, 2), \ldots, T(M, N)$, and $\mathbf{Q}$ is a column vector in which the values of the source term are stored. Values of the coefficients for $T(i, j)$ with $i$ and $j$ outside their respective ranges must be set to zero by using the boundary conditions. Such a system of equations is obtained from the finite difference replacement of, for example, the two dimensional biharmonic equation on a rectangular Cartesian grid. We assume that the region in which the values of $\mathbf{T}$ are sought is a rectangle, though this does not restrict the use of the method to problems defined within rectangular boundaries. A finite region of any shape can always be circumscribed by a rectangle, and the difference equations for nodal points within this rectangle, but outside the region of interest, can always be written as

$$T(i,j) = 0,$$

i.e., all the coefficients except the central one are set equal to zero. If the boundaries of the problem are piecewise rectangular, the inclusion of the boundary conditions into the difference equations for the nodal points on or close to the boundary seldom causes any difficulty, but for other less regular boundary shapes, ingenuity, and sometimes approximate representations of the conditions, have to be used.

[1] A fourth order partial differential equation is elliptic if it has no real characteristics.

For the nodal points on or adjacent to the boundaries of the region under consideration, the finite difference equations must be derived incorporating the boundary conditions. First, the finite difference equation is derived for the nodal point in terms of nodal values of points within the region of interest, and some points which lie outside the relevant region, such as the values $T(0, 1)$, $T(-1, 0)$, etc. The positions of the necessary nodes are derived by constructing image points in the boundary. For example, the point (0, 1) is the image point of (2, 1) in the point (1, 1). Fourth order equations of the Biharmonic type require two non-degenerate boundary conditions at each boundary point, and each boundary condition must consist of a linear combination of the function value and its first, second, and third normal derivatives at the point. Since the pair of equations is nondegenerate we can always derive one condition which does not involve the third derivative. The general pair of conditions is then

$$\alpha_1(i,j)\,T(i,j) + \alpha_2(i,j)\left(\frac{\partial T}{\partial n}\right)_{(i,j)} + \alpha_3(i,j)\left(\frac{\partial^2 T}{\partial n^2}\right)_{(i,j)} + \alpha_4(i,j)\left(\frac{\partial^3 T}{\partial n^3}\right)_{(i,j)} = \alpha_5(i,j),$$

and

$$\beta_1(i,j)\,T(i,j) + \beta_2(i,j)\left(\frac{\partial T}{\partial n}\right)_{(i,j)} + \beta_3(i,j)\left(\frac{\partial^2 T}{\partial n^2}\right)_{(i,j)} = \beta_4(i,j).$$

Such a pair of conditions holds for all points $(i, j)$ which lie on the boundary, where $n$ is the outward normal.

Consider a boundary point $(i,j)$ for which the four points $(i-1,j)$, $(i-1,j-1)$, $(i-1,j+1)$, and $(i-2,j)$ all lie outside the region of interest. The finite difference equation for the point $(i+1,j)$ involves the value at the point $(i-1,j)$ which is outside the boundary. The finite difference form of the second boundary condition evaluated at $(i,j)$ involves the three values $T(i,j)$, $T(i-1,j)$ and $T(i+1,j)$ and is used to eliminate the value at the image point $T(i-1,j)$. Thus, the finite difference equation for the nodal value $T(i+1,j)$ now only involves values within the region of interest. The difference equation for the boundary point $(i, j)$ involves the values at all four of the above points which are positioned outside the region of interest. The finite difference form of the first boundary condition evaluated at $(i,j)$ is used to eliminate the value $T(i-2,j)$. The second boundary condition can then be used to eliminate the three remaining values which are outside the boundary by evaluating the condition at $(i,j)$, $(i,j-1)$, and $(i,j+1)$.

In the finite difference equations for the boundary points the coefficients of the values of $T$ at nodal points which lie outside the region of interest must all be set equal to zero after the evaluation of the modification to the other coefficients of internal nodal values by utilizing the boundary conditions.

For certain sets of boundary conditions the above procedure must be changed. For example, if the two boundary conditions are $T$ specified and $\partial T/\partial n$ specified,

then for the boundary nodal points we simply have $T$ = given, and this is in the form of the difference equation with twelve zero coefficients. For the points internal to the region and adjacent to the boundary the value at the one point in the thirteen point difference equation which is outside the region of interest is expressed in terms of the central value by utilizing the boundary condition specifying the normal gradient.

The matrix **A** then has a special form—all the nonzero elements lie on thirteen diagonals made up of the main diagonal and the four adjacent diagonals, two further bands of three diagonals in each of the upper and lower half triangular divisions of the matrix, and then one further diagonal in each of the upper and lower half triangles. The form of the matrix **A** is shown in Fig. 1.

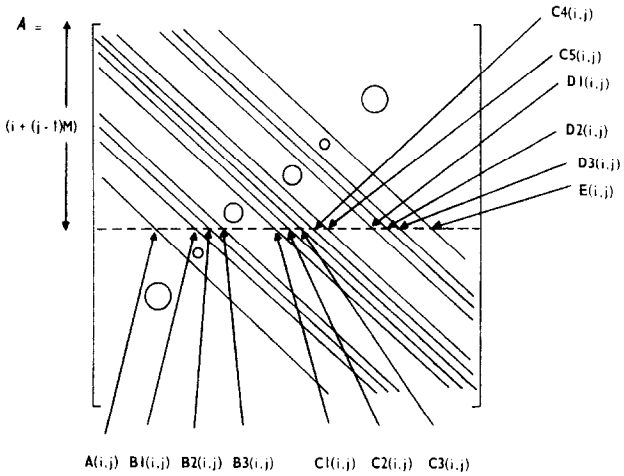


FIG. 1. The matrix A.

The method then proceeds by determining the elements of two matrices **L** and **U**, where **L** has nonzero elements on only seven diagonals, namely those diagonals corresponding to the nonzero diagonals of the matrix **A** which lie on or below the main diagonal, and where **U** has nonzero elements on seven diagonals corresponding to the nonzero diagonals of the matrix **A** which lie on or above the main diagonal. Without any loss of generality, the main diagonal elements of **U** are all set equal to unity. The product matrix **LU** is formed and has 21 nonzero diagonals, including the 13 corresponding to those of the matrix **A**, as shown in Fig. 2.

The elements coincident with those nonzero elements of **A** could be equated with those of **A** so that we would have the identity

$$\mathbf{LU} = \mathbf{A} + \mathbf{S}',$$

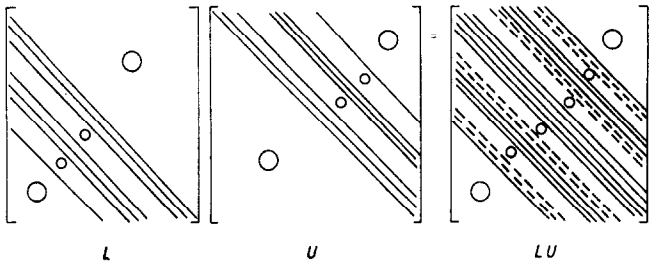


FIG. 2. The formation of the product matrix **LU**.

where the matrix $\mathbf{S}'$ has eight nonzero diagonals, namely those which are generated by the product **LU** but which are not present in the matrix **A**. Since there are 13 diagonals of the matrices **L** and **U** to be determined and there are 13 diagonals of the matrix **A** already specified, the elements of **L** and **U** are uniquely determined. The elements of $\mathbf{S}'$ could then easily be calculated.

However, the idea expounded by Stone [4] is that the matrix **LU** can be considered as the coefficient matrix of a system of 21 point difference equations. The extra terms which are present in the elements of **LUT** and which are not present in the elements of **AT** involve the values

$$T(i+1,j-2),\ T(i+2,j-2),\ T(i+2,j-1),\ T(i+3,j-1),$$
$$T(i-1,j+2),\ T(i-2,j+2),\ T(i-2,j+1),\ T(i-3,j+1).$$

We must determine all these values accurate to second order in small quantities in terms of the 13 values already present in the difference equation by using Taylor series expansions. On a uniform grid we can use

$$\begin{aligned} T(i \pm 1, j \mp 2) &= T(i, j \mp 2) + T(i \pm 1, j \mp 1) - T(i, j \mp 1) \\ &\quad + O[\Delta x\, \Delta y, \Delta x^3, \Delta y^3], \\ T(i \pm 2, j \mp 1) &= T(i \pm 2, j) + T(i \pm 1, j \mp 1) - T(i \pm 1, j) \\ &\quad + O[\Delta x\, \Delta y, \Delta x^3, \Delta y^3], \\ T(i \pm 2, j \mp 2) &= T(i, j \mp 2) + T(i \pm 2, j) - T(i, j) \\ &\quad + O[4\Delta x\, \Delta y, 8\Delta x^3, 8\Delta y^3], \end{aligned}$$

and

$$\begin{aligned} T(i \pm 3, j \mp 1) &= T(i, j \mp 1) + 3/2(T(i \pm 1, j \mp 1) - T(i \mp 1, j \mp 1)) \\ &\quad + O[9/4\Delta x^2], \end{aligned}$$

where $\Delta x$ and $\Delta y$ are the grid spacings in the $x$ and $y$ directions, respectively. These relations are accurate to second order only when the grid is uniform, unlike

those derived by Stone [4] for the five point operator system which are universally accurate, independent of the nonuniformity of the grid. (A version of the program was modified to incorporate identities similar to those given above, but derived for nonuniform grids. However, the amount of extra work involved brought no corresponding increase to the rate of convergence.) The relations detailed above are used to partially eliminate the extra terms which appear in the elements of **LUT** and not in those of **AT**. The exact amount by which they are reduced is found to critically affect the convergence rate, and a cycle of reduction parameters is found to be desirable. The parameters used for the 13 point equation solution routine are the same as those utilized by Stone [4] for the five point equation routine on a uniform grid. From the experience gained with this set of parameters, it appears that they are equally useful on nonuniform grids as on uniform grids. The maximum parameter is $\alpha_{\max}$ given by

$$\alpha_{\max} = 1 - \frac{2}{(M-1)^2 + (N-1)^2},$$

and a cycle of nine parameters is used derived from the formula

$$\alpha_m = 1 - (1 - \alpha_{\max})^{(m-1)/8}, \qquad m = 1, 2, \ldots, 9.$$

The order in which the parameters are used marginally affects the rate of convergence and since this is, in general, very rapid the use of the parameters in three sets of threes, i.e., in the order 9, 6, 3, 8, 5, 2, 7, 4, 1, is found desirable. (Notice that each set of three parameters 9, 6, 3, 8, 5, 2, and 7, 4, 1 spans a reasonably large proportion of all the parameter values.) Each parameter is used for two iterations, the first solves the equations in the order outlined above, and for the second the order in which the equations are solved is taken with $j$ decreasing from $N$ to 1. This contrivance introduces an extra degree of symmetry into the method by incorporating some different "extra" values of $T(i, j)$ on alternate iterations, and by experiment it considerably enhances the rate of convergence.

After the suggestion by Weinstein, Stone, and Kwan [5], it is suggested that the value of $\alpha_{\max}$ be varied if, for any particular problem, either the rate of convergence is slow or the procedure diverges. For the former case $\alpha_{\max}$ can be increased, but must always be less than unity, and if the method diverges, $\alpha_{\max}$ should be decreased, but must always remain positive.

The iterative scheme is derived in the following manner. We seek the solution **T** to the equation

$$\mathbf{AT} = \mathbf{Q},$$

where the elements of **A** and **Q** are known. We determine the elements of two matrices **L** and **U** both of the special forms outlined above such that

$$\mathbf{LU} = \mathbf{A} + \mathbf{S}' - \alpha\mathbf{S},$$

where the matrix $\mathbf{S}$ has nonzero elements in the same diagonals as those of $\mathbf{A}$, and it is related to the matrix $\mathbf{S}'$ defined above by using Taylor series expansions to express $\mathbf{S}'\mathbf{T}$ in terms of $\mathbf{ST}$ to second order in small quantities. Thus, we have

$$\mathbf{LUT} = \mathbf{AT} + (\mathbf{S}' - \alpha\mathbf{S})\mathbf{T},$$

where the elements of $\mathbf{L}$ and $\mathbf{U}$ have been determined in terms of those of $\mathbf{A}$ and the value of $\alpha$. The iteration scheme is then based on the successive updating equation

$$(\mathbf{A} + \mathbf{S}' - \alpha\mathbf{S})\,\mathbf{T}^{n+1} = (\mathbf{S}' - \alpha\mathbf{S})\,\mathbf{T}^{n} + \mathbf{Q},$$

i.e., given an approximation $\mathbf{T}^n$ to the function values $\mathbf{T}$, we calculate a new approximation $\mathbf{T}^{n+1}$ from the above expression. However, the formula can be simplified by subtracting $(\mathbf{A} + \mathbf{S}' - \alpha\mathbf{S})\,\mathbf{T}^n$ from both sides to obtain

$$(\mathbf{A} + \mathbf{S}' - \alpha\mathbf{S})(\mathbf{T}^{n+1} - \mathbf{T}^{n}) = \mathbf{Q} - \mathbf{AT}^{n},$$

i.e.,

$$\mathbf{LU}\delta^{n} = \mathbf{R}^{n},$$

where $\delta^n = \mathbf{T}^{n+1} - \mathbf{T}^n$ and is the correction to be made to the values $\mathbf{T}^n$ in order to obtain the new values $\mathbf{T}^{n+1}$, and $\mathbf{R}^n$ is the residual vector of the "solution" $\mathbf{T} = \mathbf{T}^n$. We note that $\mathbf{L}$ and $\mathbf{U}$ are both triangular matrices and they can, therefore, be inverted in turn in a straightforward manner, first to give

$$\mathbf{U}\delta^{n} = \mathbf{L}^{-1}\mathbf{R}^{n},$$

and then

$$\delta^{n} = \mathbf{U}^{-1}\mathbf{L}^{-1}\mathbf{R}^{n}.$$

We note that if the process converges, the sequence $\{\mathbf{T}^n\}$ has a limit, and this limit does in fact satisfy the equation

$$\mathbf{AT} = \mathbf{Q},$$

since if

$$|\,\mathbf{T}^{n+1} - \mathbf{T}^{n}\,| \to 0, \qquad \text{then} \qquad |\,\mathbf{Q} - \mathbf{AT}^{n}\,| \to 0.$$

We have, thus, defined the strongly implicit procedure for the solution of systems of 13 point difference equations.

In the next section we detail the manipulation of the equations to obtain the elements of $\mathbf{L}$ and $\mathbf{U}$, and the subsequent evaluation of the elements of the correction vector $\delta^n$. The application of this new method to a problem requiring the solution of the biharmonic equation is described in the succeeding section, and a comparison is made with another solution procedure.

## 3. The Solution Procedure

The elements of the matrices **L** and **U** are determined from those of the matrix **A** from the equation

$$\mathbf{LU} = \mathbf{A} + \mathbf{S}' - \alpha\mathbf{S},$$

where **S** is defined to make $(\mathbf{S} - \mathbf{S}')\mathbf{T}$ a small term of order $\Delta x\,\Delta y$. Some of the contribution of the additional terms which have been introduced, therefore, remain in the equation to be solved, but the precise amount varies as $\alpha$ varies namely from one pair of iterations to the next. This residue assists in the achievement of a fast iterative scheme. Not only does the value of $\alpha$ change but also the order in which the equations are solved alternates on successive iterations as explained in Section 2 and so the iteration matrix has different elements on alternate iterations. It is, therefore, necessary to recalculate the elements of the matrices **L** and **U** at the commencement of each iteration. These are determined from the set of equations

$$\begin{aligned}
A(i,j) &= a(i,j) + \alpha(a(i,j)\,c4(i,j-2) + a(i,j)\,c5(i,j-2)),\\
B1(i,j) &= a(i,j)\,d1(i,j-2) + b1(i,j) - 1.5\alpha b3(i,j)\,c5(i+1,j-1),\\
B2(i,j) &= a(i,j)\,d2(i,j-2) + b1(i,j)\,c4(i-1,j-1) + b2(i,j)\\
&\quad - \alpha(a(i,j)\,c4(i,j-2) - b3(i,j)\,c5(i+1,j-1)),\\
B3(i,j) &= a(i,j)\,d3(i,j-2) + b1(i,j)\,c5(i-1,j-1) + b2(i,j)\,c4(i,j-1)\\
&\quad + b3(i,j) + \alpha(a(i,j)\,c4(i,j-2) + b2(i,j)\,c5(i,j-1)\\
&\quad + b3(i,j)\,c4(i+1,j-1) + 1.5b3(i,j)\,c5(i+1,j-1)),\\
C1(i,j) &= b1(i,j)\,d1(i-1,j-1) + c1(i,j) + \alpha(c1(i,j)\,e(i-2,j)\\
&\quad + c1(i,j)\,d2(i-2,j) + c2(i,j)\,d1(i-1,j)),\\
C2(i,j) &= b1(i,j)\,d2(i-1,j-1) + b2(i,j)\,d1(i,j-1) + c1(i,j)\,c4(i-2,j)\\
&\quad + c2(i,j) - \alpha(c1(i,j)\,d2(i-2,j) + c2(i,j)\,d1(i-1,j)),\\
C3(i,j) &= b1(i,j)\,d3(i-1,j-1) + b2(i,j)\,d2(i,j-1) + b3(i,j)\,d1(i+1,j-1)\\
&\quad + c1(i,j)\,c5(i-2,j) + c3(i,j) + c2(i,j)\,c4(i-1,j)\\
&\quad + a(i,j)\,e(i,j-2) - \alpha(c1(i,j)\,e(i-2,j) + a(i,j)\,c5(i,j-2)),\\
C4(i,j) &= b2(i,j)\,d3(i,j-1) + b3(i,j)\,d2(i+1,j-1) + c2(i,j)\,c5(i-1,j)\\
&\quad + c3(i,j)\,c4(i,j) - \alpha(b2(i,j)\,c5(i,j-1) + b3(i,j)\,c4(i+1,j-1)),\\
C5(i,j) &= b3(i,j)\,d3(i+1,j-1) + c3(i,j)\,c5(i,j) + \alpha(b2(i,j)\,c5(i,j-1)\\
&\quad + b3(i,j)\,c4(i+1,j-1) + a(i,j)\,c5(i,j-2)),
\end{aligned}$$

$$
\begin{aligned}
D1(i,j) = {} & c1(i,j)\,d3(i-2,j) + c2(i,j)\,d2(i-1,j) + c3(i,j)\,d1(i,j) \\
& + b1(i,j)\,e(i-1,j-1) + \alpha(c1(i,j)\,d2(i-2,j) + c2(i,j)\,d1(i-1,j) \\
& + c2(i,j)\,e(i-1,j) + 1.5c1(i,j)\,d1(i-2,j)),
\end{aligned}
$$

$$
\begin{aligned}
D2(i,j) = {} & c2(i,j)\,d3(i-1,j) + c3(i,j)\,d2(i,j) + b2(i,j)\,e(i,j-1) \\
& + \alpha(c1(i,j)\,d1(i-2,j) - c2(i,j)\,e(i-1,j)),
\end{aligned}
$$

$$
D3(i,j) = c3(i,j)\,d3(i,j) + b3(i,j)\,e(i+1,j-1) - 1.5\alpha c1(i,j)\,d1(i-2,j),
$$

and

$$
E(i,j) = c3(i,j)\,e(i,j) + \alpha(c1(i,j)\,e(i-2,j) + c2(i,j)\,e(i-1,j)),
$$

where $\alpha$ is the convergence parameter. These equations are rearranged to express the values of $a(i,j)$, $b1(i,j)$,..., $e(i,j)$ in terms of the values of the coefficients of the matrix $\mathbf{A}$, namely $A(i,j)$, $B1(i,j)$,..., $E(i,j)$, and values of $a(i,j)$, $b1(i,j)$,..., $e(i,j)$ which are already known.

On alternate iterations, the order of solution of the difference equations has $j$ decreasing from $N$ to 1 in place of $j$ increasing from 1 to $N$. This alters the values of the coefficients $A(i,j)$, $B1(i,j)$, etc. in the above equations, but little extra work is involved, and this modification is accompanied by a considerable increase in the rate of convergence.

Having determined the values of the elements of the matrices $\mathbf{L}$ and $\mathbf{U}$ we solve the matrix equation

$$
\mathbf{LU}\delta^n = \mathbf{R}^n,
$$

where $\mathbf{R}^n = \mathbf{Q} - \mathbf{AT}^n$. We first solve for $\mathbf{V}$ the system

$$
\mathbf{LV} = \mathbf{R}^n,
$$

i.e.,

$$
\mathbf{V} = \mathbf{L}^{-1}\mathbf{R}^n,
$$

where

$$
\mathbf{V} = \mathbf{U}\delta^n.
$$

With

$$
\begin{aligned}
R^n(i,j) = {} & Q(i,j) - A(i,j)\,T^n(i,j-2) - B1(i,j)\,T^n(i-1,j-1) \\
& - B2(i,j)\,T^n(i,j-1) - B3(i,j)\,T^n(i+1,j-1) - C1(i,j)\,T^n(i-2,j) \\
& - C2(i,j)\,T^n(i-1,j) - C3(i,j)\,T^n(i,j) - C4(i,j)\,T^n(i+1,j) \\
& - C5(i,j)\,T^n(i+2,j) - D1(i,j)\,T^n(i-1,j+1) - D2(i,j)\,T^n(i,j+1) \\
& - D3(i,j)\,T^n(i+1,j+1) - E(i,j)\,T^n(i,j+2),
\end{aligned}
$$

the elements of **V** are determined from the equations

$$a(i,j)\,V(i,j-2) + b1(i,j)\,V(i-1,j-1) + b2(i,j)\,V(i,j-1) + b3(i,j)\,V(i+1,j-1) + c1(i,j)\,V(i-2,j) + c2(i,j)\,V(i-1,j) + c3(i,j)\,V(i,j) = R^n(i,j).$$

Having determined all the elements of the matrix **U** and the elements of **V**, we can determine the elements of $\boldsymbol{\delta}^n$ from the equation

$$\mathbf{U}\boldsymbol{\delta}^n = \mathbf{V},$$

namely

$$\delta^n(i,j) + c4(i,j)\,\delta^n(i+1,j) + c5(i,j)\,\delta^n(i+2,j) + d1(i,j)\,\delta^n(i-1,j+1) + d2(i,j)\,\delta^n(i,j+1) + d3(i,j)\,\delta^n(i+1,j+1) + e(i,j)\,\delta^n(i,j+2) = V(i,j).$$

The latter equations are solved in the reverse order, namely $i$ and $j$ are taken in the reverse order in which the equations were numbered for the particular iteration.

As mentioned above, it is necessary to store all the nonzero elements of the matrix **U** and all the elements of the vectors $\boldsymbol{\delta}^n$ and **V**.

## 4. A Sample Problem

A test problem was run to compare the strongly implicit procedure for solving the biharmonic equation with the "fast finite difference method" detailed by Greenspan and Schultz [2] in a recent paper. Their method consisted of factorizing the biharmonic equation into two second order elliptic equations, one Laplace equation and one Poisson equation. Namely, decompose the equation

$$\nabla^4 T = 0,$$

into the two equations

$$w = \nabla^2 T,$$

with

$$\nabla^2 w = 0.$$

They then used successive over relaxation (S.O.R.) to converge each of these second order equations using the usual five point difference replacement of the Laplacian. They found it necessary to smooth (or under relax) the solutions obtained by each of these inner iterations prior to proceeding to the determination of a new approximation for the other variable. Greenspan and Schultz [2] quoted times required to solve several problems using a UNIVAC 1108, but since such a computer was not available to the present author, a program was coded using their method for use on the IBM370/165.

The problem solved is the determination of a function $T$ which satisfies the biharmonic equation in the unit square region $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, subject to boundary conditions on $T$ and on $\partial T/\partial n$, where $n$ is the outward pointing normal, at all the boundary points. The boundary conditions were derived from the function

$$T = x^3 - 3y^2 + 2xy,$$

which satisfies the biharmonic equation, and with which the solutions obtained can be compared.

The method of Greenspan and Schultz requires the use of four parameters, namely one relaxation parameter for each of the two inner iterative procedures using S.O.R. on $T$ and $w$, respectively, and two smoothing parameters with which to under relax the solutions obtained by the inner iterations on $T$ and $w$ before proceeding to the inner iteration on the other variable. Following Greenspan and Schultz, a $21 \times 21$ uniform grid was used, and the same parameters as detailed by them were used. Unfortunately, the authors did not state the level of convergence to which they "solved" the inner iterations using S.O.R., and although several degrees of convergence were tried, we were unable to match exactly the case they described. Generally speaking, the more accurately one determines the solutions to the inner iterations, the fewer outer iterations are required. By using convergence criteria on the inner iterations calculated in such a manner that convergence was achieved to the whole problem when only one inner iteration was required for each of $T$ and $w$, a total of 45 outer iterations on the whole system were required (a maximum of two hundred inner iterations was specified). With less strict convergence criteria on the inner iterations, in excess of 45 outer iterations were required, until in the extreme limit when only one inner iteration on each of $T$ and $w$ was used, a total of 441 outer iterations was required. However, for all the cases run, the total running time of the programs was between 4.9 and 5.5 sec (the programs were written in FORTRAN and compiled with the $H$ compiler using OPT = 2.)

The same problem was solved utilizing a subroutine which was written to solve systems of thirteen point difference equations based on the solution procedure outlined in Sections 2 and 3. Using the $21 \times 21$ uniform grid, the coefficients of the difference equations for the internal $17 \times 17$ nodal point square are the usual biharmonic replacements namely

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1 \\ & 2 & -8 & 2 & \\ & & 1 & & \end{array}$$

For the boundary points, the value of the function is specified to be equal to the

value of the analytical function $x^3 - 3y^2 + 2xy$; thus, the difference equations for the boundary points are simply $T =$ given. The 13 point difference molecule for points adjacent to the boundary involves the value of the function at a point which is outside the square region of interest. However, since the value of the normal derivative at boundary nodes can be calculated from the analytical function, the value of the function at the image points outside the region of interest can be expressed in terms of the value of the function at a nodal point within the region of interest. Thus we derive a system of finite difference equations of the form specified in Section 2. The solution procedure described in that section was then used with the standard set of iteration parameters to determine successive approximations to the solution. On the first iteration the equations are solved in the original order and the elements of the matrices **L** and **U** are first determined using the equations given in Section 2. The value of the correction vector $\delta^1$ is then determined, and, hence, the first approximation to the solution is derived. On the next iteration the difference equations are solved with $j$ decreasing from $N$ to 1 and $i$ increasing from 1 to $M$, otherwise the details are the same. After each pair of iterations a new iteration parameter is calculated and then used.

Convergence was achieved in 10 iterations which took 1.4 sec of computing time. This means that convergence was achieved in about one quarter to one third the amount of time required by the previous program. The results obtained by the two methods of solution were of very similar accuracies.

From the experience gained by using the strongly implicit procedures, it is believed that they often require considerably less computation than alternative iterative methods, and, even more significant, the standard set of parameters provide rapid rates of convergence for almost all the problems tried to date, thus eliminating the necessity to optimize a parameter or a set of parameters as is so often necessary with other iterative schemes. For highly nonhomogeneous problems, such as those derived on very nonuniform meshes, it is sometimes necessary to reduce the value of $\alpha_{\max}$, and the rates of convergence are less rapid than those obtained for more homogeneous problems. Nevertheless, even in these cases the strongly implicit procedure always seems to have better convergence rates than other schemes.

## 5. Other Applications

The biharmonic operator appears as the dominant factor in many equations of mathematical physics. For example, the vibrations of a thin plate are determined by the solutions to the time dependent equation

$$\frac{\partial^2 u}{\partial t^2} = \nabla^4 u,$$

where $u$ is the displacement of the plate from its equilibrium position.

The slow flow of a viscous incompressible fluid in two dimensions is governed by the fourth order nonlinear equation

$$\nabla^4\psi - R\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi) - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi)\right) = 0,$$

where $\psi$ is the stream function of the fluid defined by the relations

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x},$$

where $u$ and $v$ are the velocity components in the $x$ and $y$ directions, respectively. The strongly implicit procedure has been used to solve this equation by writing it in the form

$$\nabla^4\psi^{n+1} - R\left(\frac{\partial\psi^n}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi^{n+1}) - \frac{\partial\psi^n}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi^{n+1})\right) = 0,$$

where $\psi^n$ is the $n$th approximation obtained for $\psi$ and the set of difference equations derived from the differential equation are then solved for the values of $\psi^{n+1}$. For a given mesh provided the value of $R$ was sufficiently small, the test problem which was the flow in a square cavity as considered in Ref. [1] and [3] was amenable to solution. However, if space centered differences are used for all the derivatives, for any given mesh interval there is a related Reynolds number $R$ above which solutions could not be obtained because the method of solution becomes unstable. The results obtained at Reynolds numbers below the critical values are in very close agreement with those of Burggraf [1].

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